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Approximate Interpolation by Functions in a Haar Space

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Let $\{\varphi_1, ..., \varphi_n\}$ be a Chebyshev system spanning an n-dimensional Haar subspace H of $C(I)$, where I is a proper compact interval in $\mathbb{R}$. As is well known, for each a in $C(I)$ there exists an element h of H that interpolates a on a given set of n distinct points of I. In [4, (2,2)], I proved that this interpolation can be performed approximately, with error at most ε, on a set of $n+1$ distinct points of I, provided that set is sufficiently close to degenerating into an n-point one.[1] The purpose of this note is to show how that approximate interpolation result can be substantially improved.

As in my earlier papers on Chebyshev approximation (for example, [3, 4, 5]), we shall work entirely within the constructive framework erected by the late Errett Bishop [1, 2]. To do so successfully, we must pay particular attention to the interpretation of definitions and propositions. Thus, for example, we define a *Chebyshev system* over a proper compact interval I to be a set $\{\varphi_1, ..., \varphi_n\}$ of elements of $C(I)$ such that $\beta(\alpha) > 0$ for each α in $[0, l/n]$, where l is the length of I and

$$\begin{aligned} \beta(\alpha) &= \inf\{|\varphi_1(x)| : x \in I\} && \text{if } n = 1, \\ &= \inf\{|\det[\varphi_j(x_i)]| : x_1, ..., x_n \in I, \min_{1 \leqslant i < j \leqslant n} |x_i - x_j| \geqslant \alpha\} && \text{if } n \geqslant 2. \end{aligned}$$

Also, a *modulus of uniform continuity* for a mapping f between metric spaces (X, ρ) and (X', ρ') is a function $\omega: \mathbb{R}^+ \to \mathbb{R}^+$ such that if $\varepsilon > 0$ and $\rho(x, y) < \omega(\varepsilon)$, then $\rho'(f(x), f(y)) < \varepsilon$.

[1] As it stands, Proposition (2.2) in [4] is wrong in stating that the approximate interpolating function can be chosen to have a modulus of continuity independent of ε. I am grateful to the referee for pointing out this error, and for making other helpful comments about this paper.

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In the remainder of this paper, $\{\varphi_1, ..., \varphi_n\}$ will be a fixed Chebyshev system, generating a Haar space H, over the compact interval I of length l. Also, ρ will denote the Hausdorff metric on the set of compact subsets of I.

The following lemmas prepare us for the proof of our main result.

LEMMA 1. *If A is a totally bounded subset of $C(I)$, then the map $(a, x) \to a(x)$ is uniformly continuous on $A \times I$.*

Proof. Since A is totally bounded, it is equicontinuous; let ω be a common modulus of uniform continuity for the functions in A. Given $\varepsilon > 0$, consider $a, b \in A$ and $x, y \in I$ such that $\|a - b\| < \varepsilon$ and $|x - y| < \omega(\varepsilon)$. We have

$$|a(x) - b(y)| \leqslant |a(x) - b(x)| + |b(x) - b(y)| \leqslant \varepsilon + \varepsilon = 2\varepsilon,$$

from which, as ε is arbitrary, the result follows. ∎

For each α with $0 < \alpha \leqslant l/n$, let

$$J_\alpha \equiv \{(x_1, ..., x_n) \in I^n : x_{i+1} - x_i \geqslant \alpha \text{ for all } i\ (1 \leqslant i \leqslant n-1)\},$$

a compact subset of I^n. For each $a \in C(I)$ and each $\mathbf{x} \equiv (x_1, ..., x_n) \in J_\alpha$, let $\theta(a, \mathbf{x})$ be the unique element h of H such that $h(x_i) = a(x_i)$ for each i $(1 \leqslant i \leqslant n)$.

LEMMA 2. *If A is a totally bounded subset of $C(I)$, then the map θ is uniformly continuous on $A \times J_\alpha$ for each α in $(0, l/n]$.*

Proof. Let $0 < \alpha \leqslant l/n$. By Cramer's Rule, for each $\alpha \in A$ and each $\mathbf{x} \equiv (x_1, ..., x_n) \in J_\alpha$,

$$\theta(a, \mathbf{x}) = \sum_{j=1}^{n} (\Delta(\mathbf{x})^{-1} \sum_{i=1}^{n} a(x_i) \varphi_{ij}(\mathbf{x})) \varphi_j,$$

where $\Delta(\mathbf{x})$ is the determinant of the n-by-n matrix $M \equiv [\varphi_j(x_i)]$, and $\varphi_{ij}(\mathbf{x})$ is the cofactor of $\varphi_j(x_i)$ in M. By the Haar condition, Δ is bounded away from 0 on J_α; whence the map $\mathbf{x} \to \Delta(\mathbf{x})^{-1}$ is uniformly continuous on J_α. The cofactor map $\mathbf{x} \to \varphi_{ij}(\mathbf{x})$ is certainly uniformly continuous, and hence bounded, on J_α. Also, for all $a, b \in A$ and $\mathbf{x}, \mathbf{y} \in J_\alpha$, we have

$$\begin{aligned} |a(x_i)\varphi_{ij}(\mathbf{x}) - b(y_i)\varphi_{ij}(\mathbf{y})| &\leqslant |a(x_i) - b(x_i)|\ |\varphi_{ij}(\mathbf{x})| + |b(x_i)|\ |\varphi_{ij}(\mathbf{x}) - \varphi_{ij}(\mathbf{y})| \\ &\quad + |\varphi_{ij}(\mathbf{y})|\ |b(x_i) - b(y_i)| \\ &\leqslant c(\|a - b\| + |\varphi_{ij}(\mathbf{x}) - \varphi_{ij}(\mathbf{y})| + |b(x_i) - b(y_i)|), \end{aligned}$$

where $c > 0$ is a bound for $|\varphi_{ij}|$ on J_α and for all the functions in A. Since A is equicontinuous, we now see that the maps $(a, \mathbf{x}) \to \Delta(\mathbf{x})^{-1} a(x_i)\varphi_{ij}(\mathbf{x})$ $(1 \leqslant i, j \leqslant n)$ have a common modulus of uniform continuity on $A \times J_\alpha$. The desired result follows almost immediately. ∎

LEMMA 3. *Let X be an equicontinuous subset of $C(I)$, ω a common modulus of uniform continuity for the functions in X, S a subset of I, and a, h elements of X such that $|(a-h)(s)| < \varepsilon/3$ for all s in S. Then $|(a-h)(x)| < \varepsilon$ whenever $x \in I$ and $\rho(x, S) < \omega(\varepsilon/3)$.*

Proof. If $x \in I$ *and* $\rho(x, S) < \omega(\varepsilon/3)$, then, choosing s in S with $|x - s| < \omega(\varepsilon/3)$, we have

$$\begin{aligned}|(a-h)(x)| &\leqslant |a(x) - a(s)| + |(a-h)(s)| + |h(s) - h(x)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \quad ∎\end{aligned}$$

Our next lemma strengthens Proposition (2.2) of [4].

LEMMA 4. *Let A be a totally bounded subset of $C(I)$, and m an integer with $1 \leqslant m \leqslant n$. Then for each $\varepsilon > 0$ there exist $\delta_m > 0$ and a totally bounded subset B_m of H with the following property: if $a \in A$ and $x_1, ..., x_m \in I$, then there exists h in B_m such that $|(a-h)(x)| < \varepsilon$ whenever $x \in I$ and $\rho(x, x_i) < \delta_m$ for some i.*

Proof. We proceed by induction on m. If $m = 1$, use [3, (2.7)] to construct ψ in H such that $\inf\{\psi(x): x \in I\} > 0$; then $1/\psi$ is uniformly continuous on I. It follows from this and Lemma 1 that the map $(a, x) \to a(x)\psi(x)^{-1}\psi$ is uniformly continuous on the totally bounded set $A \times I$; whence

$$B_1 \equiv \{a(x)\psi(x)^{-1}\psi: a \in A, x \in I\}$$

is totally bounded and therefore equicontinuous. Let ω be a common modulus of uniform continuity for the functions in $A \cup B_1$, let $\varepsilon > 0$, and write $\delta_1 \equiv \omega(\varepsilon/3)$. Consider $a \in A$ and $x_1 \in I$. With $h \equiv a(x_1)\psi(x_1)^{-1}\psi \in B_1$, we have $h(x_1) = a(x_1)$; whence, by Lemma 3, $|(a-h)(x)| < \varepsilon$ whenever $x \in I$ and $\rho(x, x_1) < \delta_1$. This completes the proof for $m = 1$.

Now let $k \in \{1, ..., n-1\}$, suppose we have proved the lemma for $m = k$, and consider the case $m = k + 1$. Given $\varepsilon > 0$, by our induction hypothesis we can find a positive number $\delta < 2l$, and a totally bounded subset B of H, with the following property: if $a \in A$ and if $x_1, ..., x_k \in I$, then there exists h in B such that $|(a-h)(x)| < \varepsilon/3$ whenever $x \in I$ and $\rho(x, x_i) < \delta$ for some

$i \in \{1, ..., k\}$. Write $\alpha \equiv \delta/2n$. In view of Lemma 2 and the induction hypothesis,

$$B_{k+1} \equiv \theta(A \times J_\alpha) \cup B$$

is a totally bounded, and therefore equicontinuous, subset of H. Let ω be a common modulus of uniform continuity for the functions in $A \cup B_{k+1}$, and write $\delta_{k+1} \equiv \omega(\varepsilon/3)$. (Note that as α depends on ε, so do B_{k+1}, ω, and δ_{k+1}.) Consider $a \in A$ and $x_1, ..., x_{k+1} \in I$. In view of Lemma 3 and our choice of δ_{k+1}, it will suffice to construct h in B_{k+1} such that

$$|(a-h)(x_i)| < \varepsilon/3, \qquad \forall i \in \{1, ..., k+1\}. \tag{$*$}$$

To this end, let

$$\sigma \equiv \min\{|x_i - x_j| : 1 \leqslant i < j \leqslant k+1\}.$$

Either $2n\alpha > \sigma$ or $\sigma > n\alpha$. In the former case, we may assume that $|x_k - x_{k+1}| < 2n\alpha = \delta$. By our induction hypothesis, there exists h in B such that $|(a-h)(x)| < \varepsilon/3$ whenever $x \in I$ and $\rho(x, x_i) < \delta$ for some $i \in \{1, ..., k\}$; whence, clearly, $(*)$ holds. In the case $\sigma > n\alpha$, we can choose ξ in J_α such that $x_i \in \{\xi_1, ..., \xi_n\}$ for each $i \in \{1, ..., k+1\}$. The Haar condition then ensures that there exists h in $\theta(A \times J_\alpha)$ such that $|(a-h)(x_i)| = 0 < \varepsilon/3$ for $1 \leqslant i \leqslant k+1$. This completes the induction. ∎

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We now come to our main results, each of which is an expression of the continuity of the process whereby functions in $C(I)$ are interpolated by elements of H.

THEOREM. *If A is a totally bounded subset of $C(I)$, then for each $\varepsilon > 0$ there exist $\delta > 0$ and a totally bounded subset B of H with the following property: if $\varphi \in C(I)$, $\rho(\varphi, A) < \delta$, and $x_1, ..., x_n \in I$, then there exists $h \in B$ such that $|(\varphi - h)(x)| < \varepsilon$ whenever $x \in I$ and $\rho(x, x_i) < \delta$ for some i.*

Proof. By Lemma 4, there exist $\delta' > 0$ and a totally bounded subset B of H with the following property: if $a \in A$ and $x_1, ..., x_n \in I$, then there exists h in B such that $|(a-h)(x)| < \varepsilon/2$ whenever $x \in I$ and $\rho(x, x_i) < \delta'$ for some i. It now suffices to take $\delta \equiv \min\{\delta', \varepsilon/2\}$. ∎

We end with two immediate corollaries of our theorem. The first corollary says that under appropriate circumstances, we can interpolate $\varphi \in C(I)$ by $h \in H$ with error at most ε on a set of $2n$ points, provided that set is a union of two sets that are sufficiently close with respect to the Hausdorff metric ρ on the set of compact subsets of I; the second corollary is a special case of the first and subsumes Proposition (2.2) of [4].

COROLLARY 1. *If A is a totally bounded subset of $C(I)$, then for each $\varepsilon > 0$ there exist $\delta > 0$ and a totally bounded subset B of H with the following property: if $\varphi \in C(I)$, if $\rho(\varphi, A) < \delta$, and if $S \equiv \{x_1, ..., x_n\}$ and $S' \equiv \{x'_1, ..., x'_n\}$ are subsets of I with $\rho(S, S') < \delta$, then there exists $h \in B$ such that $|(\varphi - h)(x)| < \varepsilon$ for all x in $S \cup S'$.*

COROLLARY 2. *If A is a totally bounded subset of $C(I)$, then for each $\varepsilon > 0$ there exist $\delta > 0$ and a totally bounded subset B of H with the following property: if $\varphi \in C(I)$, $\rho(\varphi, A) < \delta$, and $x_1, ..., x_{n+1}$ are points of I such that $\min\{|x_i - x_j|: 1 \leqslant i < j \leqslant n+1\} < \delta$, then there exists h in B such that $|(\varphi - h)(x_i)| < \varepsilon$ for $i = 1, ..., n+1$.*

Since we have been rigorously constructive in the foregoing, the proofs of our results, taken together, embody an algorithm for the construction of the function h with the desired interpolation property. In fact, it would be a comparatively routine matter to translate those proofs into a (doubtless not very efficient) PASCAL program for the computation of h from the relevant data.

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